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Alternation in (Weighted) Ordinary Rational Approximation on a Subset

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Let X be a compact subset of the finite interval $[\alpha, \beta]$. For $g \in C(X)$ define

$$\| g \| = \max\{| g(x)| : x \in X\}.$$

Let H_l be the set of polynomials of degree $\leqslant l$. Let

$$R_m{}^n(X) = \{p/q : p \in H_n,\ q \in H_m,\ q(x) > 0 \text{ for } x \in X\}.$$

Let s be an element of $C(X)$, an ordinary (multiplicative) weight function. Common choices in applications are $s = 1$ (absolute error), and (in case f never vanishes) $s = 1/f$ (relative error). The approximation problem is, given $f \in C(X)$, minimize $\| s * (f - r)\|$ over $r \in R_m{}^n(X)$. A minimizing r is called a best approximation to f on X. We wish to find an (elegant and easily checked) characterization of best approximations and whether they are unique.

In the case X is an interval, this problem has a well-known classical solution in which best approximation is characterized by alternation of the error $s * (f - r)$ [0, 55; 1, 163; 3; 5, 122]. For general X, the characterizations of Cheney [1, 159–160] and the author [2, 201–202] hold, but are not easy to apply.

To avoid trivial cases, we assume that X has at least $n + m + 1$ points at which s does not vanish. There is no loss of generality in assuming that $s \geqslant 0$ (otherwise replace s by $| s |$).

DEFINITION. Let p be a polynomial $\not\equiv 0$. Then ∂p is the exact degree of p.

DEFINITION. Let $r = p/q \in R_m{}^n(X)$ be given. Let p'/q' denote an equivalent irreducible rational function (if $p = 0$, we set $q' = 1$). The *degree* of nonzero p/q, written $\rho(p/q)$, is $1 + \max\{n + \partial q', m + \partial p'\}$. The degree of 0, $\rho(0)$, is $n + 1$.

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We first prove a generalization of a result of de la Vallée-Poussin. It is useful in showing that nearly alternating approximations are near best.

LEMMA. *Let* $l = \rho(p/q)$. *Let* $s * (f - p/q)\,\mathrm{sgn}(q')$ *alternate in sign on* $\{x_0, ..., x_l\} \subseteq X$, $x_0 < \cdots < x_l$. *Then if* $r_0 = p_0/q_0$ *is a different element of* $R_{n}^{n}(x)$,

$$\max\{s(x_i) * |f(x_i) - r_0(x_i)|: i = 0, ..., l\}$$
$$> \min\{s(x_i) * |f(x_i) - r(x_i)|: i = 0, ..., l\}.$$

Proof. Suppose not. Assume without loss of generality that $q'(x_0)(f(x_0) - r(x_0)) > 0$. Then

$$q'(x_0)[r_0(x_0) - r(x_0)] \geqslant 0$$
$$q'(x_1)[r_0(x_1) - r(x_1)] \leqslant 0$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$\cdots\cdots$$

Multiply the i-th inequality by $q_0(x_i)\,q(x_i) > 0$. Let us write $q(x_i) = w(x_i)\,q'(x_i)$ so that $\mathrm{sgn}(w(x_i)) = \mathrm{sgn}(q'(x_i))$.

Define $t = p_0 q' - p' q_0$; then we have

$$t(x_0) = p_0(x_0)\,q'(x_0) - p'(x_0)\,q_0(x_0) \geqslant 0$$
$$t(x_1) = p_0(x_1)\,q'(x_1) - p'(x_1)\,q_0(x_1) \leqslant 0$$
$$\cdots\cdots$$
$$\cdots\cdots$$

t is alternately $\geqslant 0$ and $\leqslant 0$ on $l + 1$ consecutive points, hence s has l zeros, counting double zeros twice. But t is a polynomial of degree at most $l - 1$, so we have a contradiction.

DEFINITION. A function $g \in C(X)$ *alternates* l times on X if there exists $\{x_0, ..., x_l\} \subseteq X$, $x_0 < \cdots < x_l$, such that

$$|g(x_i)| = \|g\| \qquad i = 0, ..., l.$$
$$g(x_i) = (-1)^i g(x_0)$$

THEOREM. *A necessary and sufficient condition that* $r = p/q$ *be best to* f *is that* $s * (f - r)\,\mathrm{sgn}(q)$ *alternate* $\rho(p/q)$ *times on* X.

Proof. Sufficiency follows from the lemma preceding. *Necessity* Let f not be an approximant and r be best. Let

$$M(r) = \{x: s(x) * |f(x) - r(x)| = \|s * (f - r)\|\}.$$

By the bottom corollary of [2, 202], 0 is in the convex hull of $\{\sigma(x)\, s(x)\, \Phi(x)$: $x \in M(r)\}$ where $\sigma(x) = \operatorname{sgn}(f(x) - r(x))$ and $\Phi(x) = (\theta_1, \ldots, \theta_l)$, where $\theta_1, \ldots, \theta_l$ is a basis for $pQ - qP$ where $P = H_n$ and $Q = H_m$. By the arguments of the lemma of Cheney [1, 162] l is $\rho(p/q)$. Let $w(x)$ be the product of common factors of p and q. Then 0 is in the convex hull of

$$\{\sigma(x)\, s(x)\, w(x)\, \Phi'(x) : x \in M(r)\}$$

where

$$\Phi'(x) = (\theta_1', \ldots, \theta_l') \qquad \text{and} \qquad \{\theta_1', \ldots, \theta_l'\}$$

is a basis for $p'Q - q'P$. By the theorem of Caratheodory [1, 17] 0 is in the convex hull of

$$\{\sigma(x)\, s(x)\, w(x)\, \Phi'(x) : x \in Y\},$$

where Y is a subset of $M(r)$ containing at most $l + 1$ points. As $p'Q - q'P$ is a Haar subspace of dimension l [1, 162], Y has $l + 1$ points. By the lemma of Cheney [1, 74], $\sigma s w$ must alternate in sign on Y. But $\operatorname{sgn}(w(x)) = \operatorname{sgn}(q'(x))$ for $x \in X$.

If f is an approximant r, necessity of alternation is trivial.

Uniqueness follows from the theorem and preceding lemma. Alternately, it can be deduced from Cheney's unicity theorem [1, 164] and the arguments of his lemma [1, 162].

The strong unicity theory of Cheney [1, 165] holds. Thus his corollary [1, 166] applies when we replace $R_m^n[a, b]$ by $R_m^n(X)$ in the case $s > 0$.

In case best r reduced to lowest terms has a denominator which is >0 on X, $s * (f - r)$ alternates $\rho(r)$ times on X. In particular in the case best r is non-degenerate, $s * (f - r)$ alternates $n + m + 1$ times on X and the Remez algorithm can be used.

In the case $n = 0$ or $m = 0$ or $m = 1$, q' is of constant sign on X, hence we have alternation of $s * (f - r)$ (r best) $\rho(r)$ times on X. If $m \geqslant 2$ such may not be the case.

EXAMPLE. Let X be a closed subset of $[-1, 1]$ not including zero. Let $n = 1$ and $m = 2$ and

$$r(x) = p(x)/q(x) = x/x^2 = 1/x = p'(x)/q'(x).$$

If $\operatorname{sgn}(x)(f - r)$ alternates $\rho(p/q) = 1 + \max\{1 + 1, 2 + 0\} = 3$ times on X, p/q is best. Consider in particular the case $e \neq 0$ and

$$\begin{aligned} f(-1) &= r(-1) + e \\ f(-1/2) &= r(-1/2) - e \\ f(1/2) &= r(1/2) - e \\ f(1) &= r(1) + e \\ |f(x) - r(x)| &< e, \qquad \text{otherwise.} \end{aligned}$$

$\operatorname{sgn}(x)(f - r)$ alternates three times on X but $f - r$ alternates only two times on X.

In [4] Lee and Roberts consider rational approximation on discrete X. Their paper explicitly considers denominators >0 on X and gives an alternation theorem, which they attribute to Rivlin [5, p. 131], which is the exact analogue of the classical result for intervals. A difficulty in applying the theorem is that if we cancel common factors from numerator and denominator, the reduced denominator may no longer be >0 on X, as in the preceding example. The preceding example and theory shows that the reduced denominator q' must be considered if we want an alternating characterization. A check of Rivlin's theory shows he assumed denominators >0 on $[\alpha, \beta]$.

We say that r (best) has *standard alternation* if $f - r$ alternates $\rho(r)$ times on X.

It might be thought that for fixed f and for X sufficiently dense [1. 84] in the interval, that standard alternation of the error of the best approximation will occur. Such is not necessarily the case.

EXAMPLE. We will show that there exists $f \in C[-1, 1]$ and a sequence of closed subsets $\{X_k\} \to [-1, 1]$ such that the unique best approximation by $R_2^1(X_k)$ on X_k does not have standard alternation and has a pole in $[-1, 1]$. Let T_j be the j-th Chebyshev polynomial on $[-1, 1]$, defined in [1; 5] and many other texts. T_j alternates exactly j times on $[-1, 1]$ with amplitude 1. Let z be a fixed zero of T_j, say the first one left of zero. Define

$$f(x) = \operatorname{sgn}(x - z)\, T_j(x)$$
$$r_k(x) = 1/[k(x - z)] = (x - z)/[k(x - z)^2]$$
$$X_k = \{x: |f(x) - r_k(x)| \leqslant 1 - 1/k,\ x \in [-1, 1] \sim (z - 2/k, z + 2/k)\}.$$

Let fixed $x \in [-1, 1]$ be not equal to z or an extremum of T_j, then $|f(x)| < 1$. For k sufficiently large, $x \notin (z - 2/k, z + 2/k)$. For k sufficiently large.

$$|f(x) - r_k(x)| \to |f(x)| < 1.$$

Hence for all k sufficiently large, $x \in X_k$. Let x be an extremum of T_j. There exists x_k near x at which $|f - r_k|$ attains $1 - 1/k$. We claim $\{x_k\} \to x$. Suppose not, then we can assume without loss of generality that $\{x_k\} \to y \neq x$. Then $|f(y)| < 1$ as T_j has only $j + 1$ extrema on $[-1, 1]$, and since r_k converges uniformly to zero on a neighborhood of y, $|f(x_k) - r_k(x_k)| \not\to 1$. This contradicts choice of x_k. Let $x_k = z + 2/k$, then $r_k(x_k) = 1/2$ and since $f(x_k) \to 0$, $f(x_k) - r_k(x_k) \to 1/2$. Hence $x_k \in X_k$ and $\{x_k\} \to z$. Thus $\{X_k\} \to [-1, 1]$. For $x \notin (z - z/k, z + 2/k)$, $|r_k(x)| \leqslant 1/2$ and for x not close to z, $r_k(x)$ is close to zero. From this and the fact that $\operatorname{sgn}(x - z) f(x) = T_j$ alternates j times on $[-1, 1]$, it can be seen that $\operatorname{sgn}(x - z)(f - r_k)$ alternates

j times on X_k with amplitude $1 - 1/k$, hence by our characterization theorem, r_k is uniquely best in $R_2^1(X_k)$ to f on X_k for $j \geqslant 3$. Now f alternates exactly $j - 1$ times on $[-1, 1]$, from which it can be deduced by similar arguments that $f - r_k$ alternates exactly $j - 1$ times on X_k. But for standard alternation, at least three alternations of $f - r_k$ are required. Thus we do not have standard alternation in the case $j = 3$.

The set X_k of the example is infinite. It can be replaced by a finite set Y_k containing the extrema of $|f - r_k|$ on X_k and with density $1/2^k$ in X_k.

The previous example is relevant to discretization, an important result concerning which is Theorem 2 of [6]. The example shows that admissibility on $X = [-1, 1]$ of best approximations on $\{X_k\} \to [-1, 1]$ need not hold if we drop the representation hypothesis of that theorem.

The example can be extended to approximation by rationals of higher degree. Consider approximation by $R_{2+i}^{1+i}(X_k)$: in this family r_k is of degree $3 + i$. Select $j \geqslant 3 + i$ and the example goes through. For $j = 3 + i$ we do not have standard alternation.

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